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A symplectic realization of the Volterra lattice

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Abstract
We examine the multiple Hamiltonian structure and construct a symplectic realization of the Volterra model. We rediscover the hierarchy of invariants, Poisson brackets and master symmetries via the use of a recursion operator. The rational Volterra bracket is obtained using a negative recursion operator.

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1. Introduction

The Volterra model, also known as the KM system, is defined by

$$\dot{u}_i = u_i(u_{i+1} - u_{i-1}) \qquad i = 1, 2, \ldots, n, \tag{1}$$

where $u_0 = u_{n+1} = 0$. It was studied originally by Volterra in [17] to describe population evolution in a hierarchical system of competing species. It was first solved by Kac and van Moerbeke in [13], using a discrete version of inverse scattering due to Flaschka [10]. In [15], Moser gave a solution of the system using the method of continued fractions and in the process he constructed action-angle coordinates. Equations (1) can be considered as a finite-dimensional approximation of the Korteweg–de Vries (KdV) equation. They also appear in the discretization of conformal field theory; the Poisson bracket for this system can be thought as a lattice generalization of the Virasoro algebra [8]. The variables u_i are an intermediate step in the construction of the action-angle variables for the Liouville model on the lattice. This system has a close connection with the Toda lattice,

$$\begin{aligned} \dot{a}_i &= a_i(b_{i+1} - b_i) \qquad & i &= 1, \ldots, n-1 \\ \dot{b}_i &= 2\left(a_i^2 - a_{i-1}^2\right) \qquad & i &= 1, \ldots, n. \end{aligned}$$

In fact, a transformation of Hénon connects the two systems:

$$\begin{aligned} a_i &= -\tfrac{1}{2}\sqrt{u_{2i}u_{2i-1}} \qquad & i &= 1, \ldots, n-1 \\ b_i &= \tfrac{1}{2}(u_{2i-1} + u_{2i-2}) \qquad & i &= 1, \ldots, n. \end{aligned}$$

We note that the number of variables for the Toda lattice is odd and therefore we restrict our attention to the Volterra system with an odd number of variables. The Volterra system is usually associated with a simple Lie algebra of type A_n. Bogoyavlensky generalized this system for each simple Lie algebra and showed that the corresponding systems are also integrable. See [1, 2] for more details. The relation between Volterra and Toda systems is also examined in [6].

The Hamiltonian description of system (1) can be found in [7] and [3]. We will follow [3] and use the Lax pair of that reference. The Lax pair is given by

$$\dot{L} = [B, L],$$

where

$$L = \begin{pmatrix} u_1 & 0 & \sqrt{u_1u_2} & 0 & \cdots & & 0 \\ 0 & u_1+u_2 & 0 & \sqrt{u_2u_3} & & & \vdots \\ \sqrt{u_1u_2} & 0 & u_2+u_3 & & \ddots & & \\ 0 & \sqrt{u_2u_3} & & & & & \\ \vdots & & \cdots & & & & \sqrt{u_{n-1}u_n} \\ & & & & & u_{n-1}+u_n & 0 \\ & & & & \sqrt{u_{n-1}u_n} & 0 & u_n \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & \frac{1}{2}\sqrt{u_1u_2} & 0 & \cdots & & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{u_2u_3} & & & \vdots \\ -\frac{1}{2}\sqrt{u_1u_2} & 0 & 0 & & \ddots & & \\ 0 & -\frac{1}{2}\sqrt{u_2u_3} & & & & & \\ \vdots & & \cdots & & & & \frac{1}{2}\sqrt{u_{n-1}u_n} \\ & & & & & 0 & 0 \\ & & & & -\frac{1}{2}\sqrt{u_{n-1}u_n} & 0 & 0 \end{pmatrix}.$$

This is an example of an isospectral deformation; the entries of L vary over time but the eigenvalues remain constant. It follows that the functions $H_i = \frac{1}{i}\operatorname{Tr} L^i$ are constants of motion. We note that

$$H_1 = 2\sum_{i=1}^{n} u_i$$

corresponds to the total momentum and

$$H_2 = \sum_{i=1}^{n} u_i^2 + 2\sum_{i=1}^{n-1} u_i u_{i+1}$$

is the Hamiltonian.

Following [3] we define the following quadratic Poisson bracket,

$$\{u_i, u_{i+1}\} = u_i u_{i+1},$$

and all other brackets equal to zero. We denote this bracket by π_2. For this bracket det L is a Casimir and the eigenvalues of L are in involution. Of course, the functions H_i are also in involution. Taking the function $\sum_i^n u_i$ as the Hamiltonian we obtain equation (1). This bracket can be realized from the second Poisson bracket of the Toda lattice by setting the momentum variables equal to zero [7].

In [3] one also finds a cubic Poisson bracket which corresponds to the second KdV bracket in the continuum limit. It is defined by the formulae

$$\{u_i, u_{i+1}\} = u_i u_{i+1}(u_i + u_{i+1}), \qquad \{u_i, u_{i+2}\} = u_i u_{i+1} u_{i+2};$$

all other brackets are zero. We denote this bracket by π_3. In this bracket we still have involution of invariants. We also have Lenard-type relations of the form

$$\pi_3 \nabla H_i = \pi_2 \nabla H_{i+1}.$$

In [3] appears a bracket that is homogeneous of degree 1, a rational bracket constructed using a master symmetry. This bracket, denoted by π_1, has Tr L as Casimir and the Hamiltonian is $H_2 = \frac{1}{2}\operatorname{Tr} L^2$. The definition of the bracket is the following. We define the master symmetry Y_{-1} to be

$$Y_{-1} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial u_i},$$

where the f_i are determined recursively as follows:

$$f_1 = -1, \qquad f_{2i} = -\frac{u_{2i}}{u_{2i-1}} f_{2i-1}, \qquad f_{2i-1} = -f_{2i-2} - 1.$$

Taking the Lie derivative of π_2 in the direction of Y_{-1} we obtain π_1, a Poisson bracket that is homogeneous of degree 1. For $n = 5$, π_1 takes the form

$$\begin{array}{llll}
\{u_1, u_2\} = u_2 & \{u_1, u_3\} = -u_2 & \{u_1, u_4\} = \dfrac{u_2 u_4}{u_3} & \{u_1, u_5\} = -\dfrac{u_2 u_4}{u_3} \\
\{u_2, u_3\} = u_2 & \{u_2, u_4\} = -\dfrac{u_2 u_4}{u_3} & \{u_2, u_5\} = \dfrac{u_2 u_4}{u_3} & \\
\{u_3, u_4\} = u_4 & \{u_3, u_5\} = -u_4 & \{u_4, u_5\} = u_4. &
\end{array} \tag{2}$$

In this paper we rediscover this bracket using a recursion operator. The higher Poisson brackets are constructed using a sequence of master symmetries Y_i, $i = -1, 0, 1, \ldots$. We define Y_0 to be the Euler vector field

$$Y_0 = \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i}.$$

The explicit formula for Y_1 is

$$Y_1 = \sum_{i=1}^{n} U_i \frac{\partial}{\partial u_i},$$

where

$$U_i = (i+1) u_i u_{i+1} + u_i^2 + (2-i) u_{i-1} u_i.$$

It is easily checked that the bracket π_2 is obtained from π_1 by taking the Lie derivative in the direction of Y_1. Similarly, the Lie derivative of π_2 in the direction of Y_1 gives π_3.

The brackets π_1, π_2 and π_3 are just the beginning of an infinite hierarchy constructed in [3] using master symmetries. We quote the result:

Theorem 1. *There exists a sequence of Poisson tensors* π_j *and a sequence of master symmetries* Y_j *such that*

(i) π_j *are all Poisson;*
(ii) the functions H_i *are in involution with respect to all of the* π_j*;*
(iii) $Y_i(H_j) = (i + j)H_{i+j}$*;*
(iv) $L_{Y_i}\pi_j = (j - i - 2)\pi_{i+j}$*;*
(v) $[Y_i, Y_j] = (j - i)Y_{i+j}$*;*
(vi) $\pi_j \nabla H_i = \pi_{j-1}\nabla H_{i+1}$*, where* π_j *denotes the Poisson matrix of the tensor* π_j*.*

In this paper we prove the results of theorem 1 using a different approach. Namely, we construct a recursion operator in a symplectic space, define all master symmetries, invariants and Poisson brackets using results of Magri and Oevel and then project to the space of u variables.

2. Master symmetries and a theorem of Oevel

We recall the definition and basic properties of master symmetries following Fuchssteiner [12]. Consider a differential equation on a manifold M defined by a vector field χ. We are mostly interested in the case where χ is a Hamiltonian vector field. A vector field Z is a symmetry of the equation if

$$[Z, \chi] = 0.$$

A vector field Z is called a master symmetry if

$$[[Z, \chi], \chi] = 0,$$

but

$$[Z, \chi] \neq 0.$$

Master symmetries were first introduced by Fokas and Fuchssteiner in [11] in connection with the Benjamin–Ono equation.

A bi-Hamiltonian system is defined by specifying two Hamiltonian functions H_1, H_2 and two Poisson tensors π_1 and π_2, which give rise to the same Hamiltonian equations, namely, $\pi_1\nabla H_2 = \pi_2\nabla H_1$. The notion of bi-Hamiltonian structures is due to Magri [14]. Suppose that we have a bi-Hamiltonian system defined by the Poisson tensors π_1, π_2 and the Hamiltonians H_1, H_2. Assume that π_1 is symplectic. We define the recursion operator $\mathcal{R} = \pi_2\pi_1^{-1}$, the higher flows

$$\chi_i = \mathcal{R}^{i-1}\chi_1,$$

and the higher order Poisson tensors

$$\pi_i = \mathcal{R}^{i-1}\pi_1.$$

For a non-degenerate bi-Hamiltonian system, master symmetries can be generated using a method due to Oevel [16].

Theorem 2. *Suppose that* X_0 *is a conformal symmetry for both* π_1*,* π_2 *and* H_1*, i.e. for some scalars* λ*,* μ *and* ν *we have*

$$\mathcal{L}_{X_0}\pi_1 = \lambda\pi_1, \qquad \mathcal{L}_{X_0}\pi_2 = \mu\pi_2, \qquad \mathcal{L}_{X_0}H_1 = \nu H_1.$$

Then the vector fields $X_i = \mathcal{R}^i X_0$ *are master symmetries and we have*

(a) $\mathcal{L}_{X_i}H_j = (\nu + (j - 1 + i)(\mu - \lambda))H_{i+j}$*,*
(b) $\mathcal{L}_{X_i}\pi_j = (\mu + (j - i - 2)(\mu - \lambda))\pi_{i+j}$*,*
(c) $[X_i, X_j] = (\mu - \lambda)(j - i)X_{i+j}$*.*

3. Symplectic realization

We define the following transformation from $\mathbf{R}^{2n}$ to $\mathbf{R}^{2n-1}$:

$$\begin{aligned} u_{2i-1} &= -\mathrm{e}^{p_i} \qquad & i &= 1, \ldots, n, \\ u_{2i} &= \mathrm{e}^{q_{i+1}-q_i} \qquad & i &= 1, \ldots, n-1. \end{aligned} \tag{3}$$

The Hamiltonian in (q, p) coordinates is given by

$$h_1 = -\sum_{i=1}^{n} \mathrm{e}^{p_i} + \sum_{i=1}^{n-1} \mathrm{e}^{q_{i+1}-q_i}. \tag{4}$$

It is straightforward to check that Hamilton's equations for (4) correspond in the u-space to the KM system (1) via the mapping (3). The symplectic bracket in (q, p) coordinates corresponds to the quadratic bracket π_2. For this reason we will denote the standard symplectic bracket in $\mathbf{R}^{2n}$ by J_2. Our purpose is to define a bracket J_3 in $\mathbf{R}^{2n}$ which is mapped to π_3 under the transformation (3). The idea of the construction is to lift the master symmetry Y_1 from the u-space up to the (q, p)-space and obtain a vector field which we denote by X_1. The new bracket J_3 will be defined as the Lie derivative of J_2 in the direction of X_1. One possible definition for X_1 is the following:

$$X_1 = \sum_{i=1}^{n} A_i \frac{\partial}{\partial q_i} + \sum_{i=1}^{n} B_i \frac{\partial}{\partial p_i},$$

where

$$\begin{aligned} A_i &= -\mathrm{e}^{p_1} - \sum_{j=2}^{i-1} \mathrm{e}^{p_j} + (1-2\mathrm{i})\,\mathrm{e}^{p_i} + \sum_{j=1}^{i-1} \mathrm{e}^{q_{j+1}-q_j} \qquad & i &= 1, 2, \ldots, n, \\ B_i &= 2\mathrm{i}\,\mathrm{e}^{q_{i+1}-q_i} - \mathrm{e}^{p_i} + (3-2\mathrm{i})\,\mathrm{e}^{q_i-q_{i-1}} \qquad & i &= 1, 2, \ldots, n. \end{aligned}$$

We note that in the summations if an index is not defined then we ignore that whole term.

Taking the Lie derivative of the symplectic bracket J_2 in the direction of X_1 we obtain the Poisson bracket J_3,

$$\begin{aligned} \{q_i, q_j\} &= \mathrm{e}^{p_j} \qquad & 1 &\leqslant j \leqslant i-1 \leqslant n-1 \\ \{q_i, p_i\} &= -\mathrm{e}^{p_i} + \mathrm{e}^{q_i-q_{i-1}} \qquad & i &= 1, \ldots, n \\ \{q_i, p_j\} &= \mathrm{e}^{q_j-q_{j-1}} - \mathrm{e}^{q_{j+1}-q_j} \qquad & 1 &\leqslant j \leqslant i-1 \\ \{p_i, p_{i+1}\} &= \mathrm{e}^{q_{i+1}-q_i} \qquad & i &= 1, \ldots, n-1. \end{aligned} \tag{5}$$

The Jacobi identity is straightforward to check. There are four cases (three p, three q, two p one q and two q one p). Two of the cases are trivial and the other two can be broken up to at most five subcases.

Let J_2 be the symplectic bracket with Poisson matrix

$$J_2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $n \times n$ identity matrix. The bracket J_2 is mapped precisely to the bracket π_2 under transformation (3), and J_3 corresponds to π_3. We define a recursion operator as follows:

$$\mathcal{R} = J_3 J_2^{-1}.$$

This operator raises degrees and we therefore call it the positive Volterra operator. In (q, p) coordinates, the symbol χ_i is a shorthand for χ_{h_i}. It is generated, as usual, by

$$\chi_i = \mathcal{R}^{i-1} \chi_1.$$

For example,

$$h_2 = \frac{1}{2}\sum_{i=1}^{n} e^{2p_i} + \frac{1}{2}\sum_{i=1}^{n-1} e^{2(q_{i+1}-q_i)} - \sum_{i=1}^{n-1}(e^{p_i} + e^{p_{i+1}})\, e^{q_{i+1}-q_i}.$$

Note that h_2 corresponds under mapping (3) to a constant multiple of $H_2 = \frac{1}{2}\operatorname{Tr}(L)^2$. In a similar fashion we obtain the higher order Poisson tensors

$$J_i = \mathcal{R}^{i-2} J_2 \qquad i = 3, 4, \ldots.$$

We finally define the conformal symmetry

$$X_0 = \sum_{i=1}^{n} i\frac{\partial}{\partial q_i} + \sum_{i=1}^{n}\frac{\partial}{\partial p_i}.$$

The Poisson tensors J_2, J_3 and the functions h_1, h_2 define a bi-Hamiltonian pair, namely, $J_2\nabla h_2 = J_3\nabla h_1$. We note that J_3 is automatically compatible with J_2 since it is constructed using a master symmetry (see [4], p 5518). It is straightforward to verify that

$$\mathcal{L}_{X_0} J_2 = 0, \qquad \mathcal{L}_{X_0} J_3 = J_3, \qquad \mathcal{L}_{X_0}(h_1) = h_1.$$

Consequently, X_0 is a conformal symmetry for J_2, J_3 and h_1. The constants appearing in Oevel's theorem are $\lambda = 0$, $\mu = 1$ and $\nu = 1$. Therefore, we end up with the following deformation relations:

$$[X_i, h_j] = (i + j)h_{i+j}, \qquad L_{X_i} J_j = (j - i - 2)J_{i+j}, \qquad [X_i, X_j] = (j - i)X_{i+j}.$$

Projecting to the u-space under mapping (3) we obtain relations (iii)–(v) of theorem 1. Statements (i) and (ii) of theorem 1 follow easily from properties of the recursion operator.

4. The negative Volterra hierarchy

In this section we describe how the first bracket π_1 is obtained via the use of the negative operator. The negative operator was introduced in [5] in connection with the Toda lattice. We define J_1 as follows:

$$J_1 = \mathcal{N} J_2, \qquad \text{where} \quad \mathcal{N} = J_2 J_3^{-1}.$$

We then project the J_1 bracket to the u-space using transformation (3) to obtain the bracket π_1. We illustrate in detail the case $n = 5$.

We consider the Volterra model in $\mathbf{R}^6$ with coordinates $(q_1, q_2, q_3, p_1, p_2, p_3)$. Transformation (3) is given by

$$u_1 = -e^{p_1}, \qquad u_3 = -e^{p_2}, \qquad u_5 = -e^{p_3}, \qquad u_2 = e^{q_2-q_1}, \qquad u_4 = e^{q_3-q_2}. \tag{6}$$

$$J_2 = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix},$$

where I_3 is the 3×3 identity matrix, and J_3 is the Poisson matrix (5),

$$J_3 = \begin{pmatrix}
0 & -e^{p_1} & -e^{p_1} & -e^{p_1} & 0 & 0 \\
e^{p_1} & 0 & -e^{p_2} & -e^{q_2-q_1} & -e^{p_2} + e^{q_2-q_1} & 0 \\
e^{p_1} & e^{p_2} & 0 & -e^{q_2-q_1} & e^{q_2-q_1} - e^{q_3-q_2} & -e^{p_3} + e^{q_3-q_2} \\
e^{p_1} & e^{q_2-q_1} & e^{q_2-q_1} & 0 & e^{q_2-q_1} & 0 \\
0 & e^{p_2} - e^{q_2-q_1} & -e^{q_2-q_1} + e^{q_3-q_2} & -e^{q_2-q_1} & 0 & e^{q_3-q_2} \\
0 & 0 & e^{p_3} - e^{q_3-q_2} & 0 & -e^{q_3-q_2} & 0
\end{pmatrix}.$$

One can find the matrix J_1,

$$(J_1)_{1,2} = \frac{1}{D}\, e^{p_1}(e^{p_3} - e^{q_3-q_2}) \qquad (J_1)_{1,3} = \frac{1}{D}\, e^{p_1}(e^{p_2} - e^{q_3-q_2})$$

$$(J_1)_{1,4} = \frac{1}{D}\, e^{q_2-q_1}[(e^{p_3} - e^{q_3-q_2}) - e^{p_2} e^{p_3}] \qquad (J_1)_{1,5} = \frac{1}{D}\, e^{p_1} e^{q_3-q_2}$$

$$(J_1)_{1,6} = -\frac{1}{D}\, e^{p_1} e^{q_3-q_2} \qquad (J_1)_{2,3} = \frac{1}{D}\, e^{p_1} e^{p_2}$$

$$(J_1)_{2,4} = \frac{1}{D}\, e^{q_2-q_1}(e^{p_3} - e^{q_3-q_2}) \qquad (J_1)_{2,5} = \frac{1}{D}\, e^{p_1}(e^{q_3-q_2} - e^{p_3})$$

$$(J_1)_{3,4} = -\frac{1}{D}\, e^{q_2-q_1} e^{q_3-q_2} \qquad (J_1)_{3,5} = \frac{1}{D}\, e^{p_1} e^{q_3-q_2}$$

$$(J_1)_{3,6} = -\frac{1}{D}\, e^{p_1} e^{p_2} \qquad (J_1)_{4,5} = -\frac{1}{D}\, e^{p_3} e^{q_2-q_1}$$

$$(J_1)_{4,6} = \frac{1}{D}\, e^{q_2-q_1} e^{q_3-q_2} \qquad (J_1)_{5,6} = -\frac{1}{D}\, e^{p_1} e^{q_3-q_2},$$

where $D = e^{p_1} e^{p_2} e^{p_3}$. We note that D corresponds in the u-space to the square root of $\det(L)$. The projection of J_1 to the u-space under transformation (6) is precisely the bracket π_1 given in equation (2), e.g.

$$\begin{aligned}\{u_1, u_2\} &= \{-e^{p_1}, e^{q_2-q_1}\} = -e^{p_1} e^{q_2-q_1}(\{p_1, q_2\} - \{p_1, q_1\}) \\ &= -\frac{e^{p_1} e^{q_2-q_1}}{e^{p_1} e^{p_2} e^{p_3}}[e^{q_2-q_1}(-e^{p_3} + e^{q_3-q_2}) + e^{q_2-q_1} e^{p_3} - e^{q_2-q_1} e^{q_3-q_2} - e^{p_3} e^{p_2}] \\ &= e^{q_2-q_1} = u_2.\end{aligned}$$

Using the recursion operator $\mathcal{N}$ we can construct the negative Volterra hierarchy, i.e. $J_{i-1} = \mathcal{N} J_i, i = 1, 0, -1, -2, \ldots$. Using the same method of proof as in [5] one can easily show that the conclusions of theorem 1 hold for any integer value of the index. For example, for $i = 1$ we obtain a Poisson bracket J_0 which projected to the u-space gives a rational Poisson bracket of degree zero, π_0. In the case of the Volterra model in $\mathbf{R}^4$ one can find that π_0 is given by

$$\{u_1, u_2\} = \frac{u_2(u_2 + u_3)}{u_1 u_3}, \qquad \{u_3, u_1\} = \frac{u_2(u_1 + u_2 + u_3)}{u_1 u_3}, \qquad \{u_2, u_3\} = \frac{u_2(u_2 + u_1)}{u_1 u_3}.$$

5. Conclusions

This paper contains three main ingredients. The first consists of the odd-dimensional space of the Volterra model together with its multiple Hamiltonian structures. The results of this paper are not new but they are derived here using an entirely new approach. The quadratic and cubic brackets π_2 and π_3 are contained implicitly in the book of Fadeev and Takhtajan [7]. The rational, linear bracket π_1 and the rest of the hierarchy were first computed in [3] using master symmetries.

The second part is a realization of the model in a symplectic space. We define a Hamiltonian system in (q, p) coordinates, and compute master symmetries and a second Poisson structure which is used to define a bi-Hamiltonian pair. We then use the resulting recursion operator to produce the infinite hierarchy. In order to obtain a Poisson bracket that corresponds to π_1 we make use of the negative recursion operator. All the results in this part are new.

The third part is a mapping which connects the two spaces and the two systems. It is a mapping from an even $2n$ dimensional, symplectic space to an odd $(2n-1)$-dimensional space. This symplectic realization is also new. We have to mention that there is another symplectic realization of the model which goes back to Volterra. However, the map is from a $4n-2$ to a $2n-1$ space, see e.g. [9]. Due to the big difference in dimension the results of the present paper will be difficult to duplicate using that particular realization.

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